resistor; β, temperature coefficient; m, power sensitivity factor; σ, electrical conductivity, $\Omega^{-1}\cdot m^{-1}$; S, thermal potential, W/m. Indices: arc, arc; c, cathode; m, medium; t, thermistor; g, gas; ax, axial; a, anode; mm, mean mass; i, initial.

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SOME STEADY-STATE PROBLEMS IN THERMAL CONDUCTIVITY FOR A THIN METAL WIRE HEATED BY AN ELECTRICAL CURRENT

A. N. Shcherban' and V. N. Tarasevich

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Analysis is made of steady-state problems in thermal conductivity for a thin metal wire heated by an electrical current, including consideration of the temperature dependence of the thermophysical properties of the wire material and of the conditions for heat transfer with the surrounding medium.

The thermal conductivity of thin metal wires heated by an electrical current, which are widely used in the field of thermal measurement in the form of thermal sensors such as thermoanemometers, has been investigated in detail within the limitations of the linear problem where the thermophysical properties of the wire material (resistance, heat capacity, coefficient of thermal conductivity) and the conditions for heat transfer with the surrounding medium (coefficient of heat transfer) do not depend on temperature [1, 2]. Such a formulation of the problem has important theoretical significance; however, its solution is only valid for small excess temperatures and therefore can be used in practice for an extremely limited set of engineering problems.

In this paper, an analysis is made of the steady-state problems [1, 2] with consideration of a linear dependence on temperature for the resistance and for the coefficients of heat transfer and thermal conductivity of the wire.

If the heat transfer between the wire and the surrounding medium obeys Newton's law, the differential equation for steady-state thermal conductivity can be written in the form

$$\frac{d}{dx}\left[\lambda(t)\omega\frac{dt}{dx}\right] = -\frac{1}{\omega}I^2\rho(t) + \pi d\alpha(t)(t-t_b). \quad (1)$$

In first approximation, let

$$\rho(t) = \rho_0(1+\beta t);\ \lambda(t) = \lambda_0(1+\delta t);\ \alpha(t) = \alpha_0(1+\gamma t).$$

We then obtain from Eq. (1)

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$$\lambda_0\omega\frac{d}{dx}\left[(1+\delta t)\frac{dt}{dx}\right]=\pi d\alpha_0(1+\gamma t)(t-t_b)-\frac{1}{\omega}I^2\rho_0(1+\beta t). \quad (2)$$

The boundary conditions for the problem are $t_{x=0,l} = t_b$. Using the notation

$$\rho_b=\rho_0(1+\beta t_b);\ \lambda_b=\lambda_0(1+\delta t_b);\ \alpha_b=\alpha_0(1+\gamma t_b);$$

$$\beta'=\beta/(1+\beta t_b);\ \delta'=\delta/(1+\delta t_b);\ \gamma'=\gamma(1+\gamma t_b),$$

one can obtain from Eq. (2)

$$\lambda_b\omega\frac{d}{dx}\left[(1+\delta'\vartheta)\frac{d\vartheta}{dx}\right]=\pi d\alpha_b(1+\gamma'\vartheta)\vartheta-\frac{1}{\omega}I^2\rho_b(1+\beta'\vartheta). \quad (3)$$

After transformations, we obtain from Eq. (3)

$$\lambda_b\omega(1+\delta'\vartheta)\vartheta''+\lambda_b\omega\delta'\vartheta'^2=\pi d\gamma'\alpha_b\vartheta^2+\left(\pi d\alpha_b-\frac{1}{\omega}I^2\rho_b\beta'\right)\vartheta-\frac{1}{\omega}I^2\rho_b \quad (4)$$

or

$$\lambda_b\omega\delta'\theta\theta''+\lambda_b\omega\delta'\theta'^2=\pi d\gamma'\alpha_b\theta^2+\left\{\pi d\alpha_b\left(1-\frac{2\gamma'}{\delta'}\right)-\frac{1}{\omega}I^2\rho_b\beta'\right\}\theta$$
$$-\left\{\frac{1}{\omega}I^2\rho_b\left(1-\frac{\beta'}{\delta'}\right)+\frac{1}{\delta'}\pi d\alpha_b\left(1-\frac{\gamma'}{\delta'}\right)\right\}, \quad (5)$$

where $\theta=\vartheta+1/\delta'$.

Table 1 gives an analysis of particular cases of Eqs. (4) and (5) obtained above representing all the possible combinations of zero and constant values of the temperature coefficients β, δ, and γ, where some generalized function z is introduced in the table in place of the functions ϑ and θ occurring in Eqs. (4) and (5).

Table 1 makes it clear that the solution of the problem of thermal conductivity for a thin metal wire heated by an electrical current reduces to integration of the four types of ordinary differential equations with constant coefficients a, b, c, and d under the boundary conditions specified in the table.

Equation (a) has the solution of [1, 2].

We seek a solution of Eq. (b) on the basis that $d\vartheta/dx = 0$ when $x = l/2$ and $\vartheta=\vartheta_m$, where ϑ_m is as yet unknown. One can then obtain from Eq. (b)

$$\int_0^{\vartheta}\{A(\vartheta^3-\vartheta_m^3)+B(\vartheta^2-\vartheta_m^2)+C(\vartheta-\vartheta_m)\}^{-\frac{1}{2}}d\vartheta=x, \quad (6)$$

where

$$A=2\zeta/3\xi;\ B=\eta/\xi;\ C=-2a/\xi.$$

If we introduce the notation

$$h^2=2A/\{(3A\vartheta_m+B)+\sqrt{(B^2-4AC)-2AB\vartheta_m-3A^2\vartheta_m^2}\};$$
$$h'^2=2A/\{-(3A\vartheta_m+B)+\sqrt{(B^2-4AC)-2AB\vartheta_m-3A^2\vartheta_m^2}\};$$
$$\Psi=\arcsin\sqrt{1-h^2(\vartheta_m-\vartheta)};\quad \Psi_0=\arcsin\sqrt{1-h^2\vartheta_m};$$
$$k=h'(h^2+h'^2)^{-\frac{1}{2}},$$

we obtain from Eq. (6) a final solution of Eq. (b) in the form

TABLE 1. Analysis of Particular Cases of the Thermal-Conductivity Equations (4) and (5)

Temperature coefficients			Differential equation of thermal conductivity	$z\|_{x=0,l}$	Reference	Coefficients of differential equation			
β	γ	δ				ξ	ζ	η	a
0	0	0	$\xi z'' = \eta z - a$ (a)	0	[1]	$\lambda_0 \omega$	0	$\pi d \alpha_0$	$\frac{1}{\omega} I^2 \rho_0$
const	0	0			[2]			$\pi d \alpha_0 - \frac{1}{\omega} I^2 \rho_b \beta'$	$\frac{1}{\omega} I^2 \rho_b$
0	const	0	$\xi z'' = \zeta z^2 + \eta z - a$ (b)				$\pi d \alpha_b \gamma'$	$\pi d \alpha_b$	$\frac{1}{\omega} I^2 \rho_0$
const	const	0						$\pi d \alpha_b - \frac{1}{\omega} I^2 \rho_b \beta'$	$\frac{1}{\omega} I^2 \rho_b$
0	0	const	$\xi z z'' + \xi z'^2 = \eta z - a$ (c)	$\frac{1}{\delta'}$		$\lambda_b \omega \delta'$	0	$\pi d \alpha_0$	$\frac{1}{\omega} I^2 \rho_0 + \frac{1}{\delta'} \pi d \alpha_0$
const	0	const						$\pi d \alpha_0 - \frac{1}{\omega} I^2 \rho_b \beta'$	$\frac{1}{\omega} I^2 \rho_b \left(1 - \frac{\beta'}{\delta'}\right) + \frac{1}{\delta'} \pi d \alpha_0$
0	const	const	$\xi z z'' + \xi z'^2 = \zeta z^2 + \eta z - a$ (d)				$\pi d \alpha_b \gamma'$	$\pi d \alpha_0 \left(1 - \frac{2\gamma'}{\delta'}\right)$	$\frac{1}{\omega} I^2 \rho_0 + \frac{1}{\delta'} \pi d \alpha_b \left(1 - \frac{\gamma'}{\delta'}\right)$
const	const	const						$\pi d \alpha_b \left(1 - \frac{2\gamma'}{\delta'}\right) - \frac{1}{\omega} I^2 \rho_b \beta'$	$\frac{1}{\omega} I^2 \rho_b \left(1 - \frac{\beta'}{\delta'}\right) + \frac{1}{\delta'} \pi d \alpha_b \left(1 - \frac{\gamma'}{\delta'}\right)$

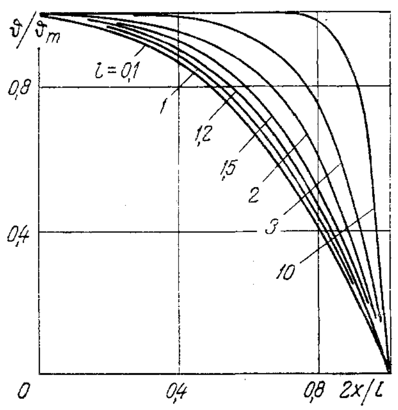

Fig. 1. Temperature distribution in a wire calculated from the equations in [1, 2].

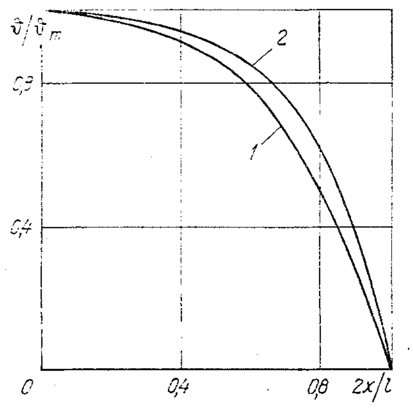

Fig. 2. Comparison of the temperature fields in a wire obtained from the equations in [1, 2] and from Eq. (7); 1) $\gamma \neq 0$; 2) 0.

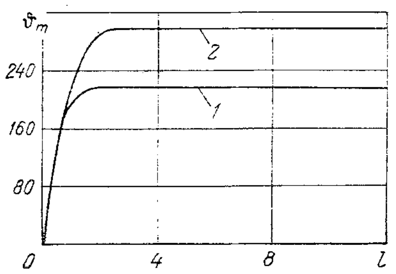

Fig. 3. Dependence of temperature in the central section of a wire on the length of the wire: 1) $\gamma = 0$; 2) $\gamma \neq 0$. ϑ_m, deg.

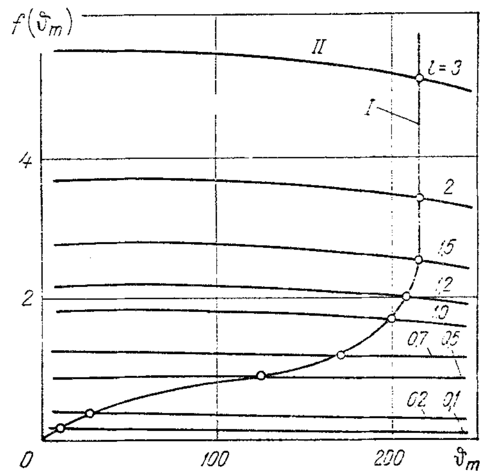

Fig. 4. Graphical solution of Eq. (9): I) $K(k) - F(k, \Psi_0)$; II) $(\sqrt{A}/4kh)l$.

$$F(k, \Psi) - F(k, \Psi_0) = \frac{x}{2hh'} \sqrt{A(h^2 + h'^2)}. \tag{7}$$

To determine the temperature drop ϑ_m in the central section of the wire, one can obtain from Eq. (6) when $x = l/2$

$$\int_0^{\vartheta_m} \{A(\vartheta^3 - \vartheta_m^3) + B(\vartheta^2 - \vartheta_m^2) + C(\vartheta - \vartheta_m)\}^{-\frac{1}{2}} d\vartheta = \frac{l}{2}, \tag{8}$$

from which we have

$$kh\{K(k) - F(k, \Psi_0)\} = \frac{\sqrt{A}}{4} l. \tag{9}$$

As an illustration, Fig. 1 gives the results of numerical calculations of the temperature of a platinum wire with d = 0.005 cm and I = 0.35 A for various values of the parameter l which were computed from the equations of [1, 2] — Eq. (a). Figure 2 shows temperature curves for the same wire and l = 1.5 cm which were computed from the equations of [1, 2] and from Eq. (7).

The dependence of the temperature in the central section of the wire on the length of the wire calculated for the cases $\gamma = 0$ and $\gamma \neq 0$ is shown in Fig. 3. In the latter case, data from the graphical solution of Eq. (9) shown in Fig. 4 were used.

Equation (c) can be transformed to

$$\theta\theta'' + \theta'^2 = A\theta - B, \tag{10}$$

where

$$A = \eta/\xi, \quad B = a/\xi,$$

from which one can write by analogy with Eqs. (6) and (8)

$$\int_0^{\theta} \theta \left[\frac{2}{3} A(\theta^3 - \theta_m^3) - B(\theta^2 - \theta_m^2) \right]^{-\frac{1}{2}} d\theta = x, \tag{11}$$

$$\int_0^{\theta_m} \theta \left[\frac{2}{3} A(\theta^3 - \theta_m^3) - B(\theta^2 - \theta_m^2) \right]^{-\frac{1}{2}} d\theta = \frac{l}{2}. \tag{12}$$

Finally, we obtain an equation for the temperature of the wire from Eq. (11):

$$\left(G + \frac{Q}{k^2}\right)\{F(k, \Psi) - F(k, \Psi_0)\} - \frac{Q}{k^2}\{E(k, \Psi) - E(k, \Psi_0)\} = \frac{x}{2}\sqrt{\frac{2}{3}A}, \tag{13}$$

and an equation for the temperature in the central section from Eq. (12):

$$\left(G + \frac{Q}{k^2}\right)\{\mathrm{K}(k) - F(k, \Psi_0)\} - \frac{Q}{k^2}\{\mathrm{E}(k) - E(k, \Psi_0)\} = \frac{l}{4}\sqrt{\frac{2}{3}A}, \tag{14}$$

where

$$h^2 = 1 \Big/ \left| \frac{3\theta_m}{2} - \frac{3B}{4A} + \frac{1}{2}\sqrt{-3\theta_m^2 + \frac{3B}{A}\theta_m + \frac{9B^2}{4A^2}} \right|;$$

$$h'^2 = 1 \Big/ \left| \frac{3\theta_m}{2} - \frac{3B}{4A} - \frac{1}{2}\sqrt{-3\theta_m^2 + \frac{3B}{A}\theta_m + \frac{9B^2}{4A^2}} \right|;$$

$$k = h'/\sqrt{h^2 + h'^2}; \quad Q = \frac{1}{h^2}\sqrt{\frac{1}{h^2} + \frac{1}{h'^2}}; \quad G = -Q(1 - h^2\theta_m);$$

$$\Psi = \arcsin\sqrt{1 - h^2(\theta_m - \theta)}; \quad \Psi_0 = \arcsin\sqrt{1 - h^2\theta_m}.$$

For the most general case, where β, δ, and $\gamma \neq 0$, Eq. (d) is brought to the form

$$\theta\theta'' + \theta'^2 = A\theta^2 + B\theta - C, \tag{15}$$

where

$$A = \zeta/\xi; \quad B = \eta/\xi; \quad C = a/\xi,$$

from which we obtain the following equations for the temperature field and for the temperature in the central section of the wire:

$$\int_{\theta_0}^{\theta} \theta \left\{ \frac{A}{2}(\theta^4 - \theta_m^4) + \frac{2}{3}B(\theta^3 - \theta_m^3) - C(\theta^2 - \theta_m^2) \right\}^{-\frac{1}{2}} d\theta = x, \tag{16}$$

$$\int_{\theta_0}^{\theta_m} \theta \left\{ \frac{A}{2}(\theta^4 - \theta_m^4) + \frac{2}{3}B(\theta^3 - \theta_m^3) - C(\theta^2 - \theta_m^2) \right\}^{-\frac{1}{2}} d\theta = \frac{l}{2}. \tag{17}$$

TABLE 2. Coefficients, Variables, and the Function $\Phi(t)$ in Eq. (18)

R	m	m'	$A(h, h')$	$B(h, h')$	k	n	t	z	$\Phi(t)$
$+1$	$-h^2$	$-h'^2$	$\frac{1}{h}$	1	$\frac{h'}{h}$	$-\frac{1}{h^2}$	$\frac{1}{h} z$	ht	$A(h, h')[\Pi(n, k, \Psi)-\Pi(n, k, \Psi_0)]$
	$-h^2$	h'^2	$-\frac{1}{\sqrt{h^2+h'^2}}$	$1-\frac{1}{h^2}$	$\frac{h'}{\sqrt{h^2+h'^2}}$	$\frac{1}{h^2-1}$	$\frac{1}{h}\sqrt{1-z^2}$	$\sqrt{1-ht}$	$\frac{A(h, h')}{B(h, h')}[\Pi(n, k, \Psi)-\Pi(n, k, \Psi_0)]$
	h^2	h'^2	$\frac{1}{h}$	1	$\frac{\sqrt{h^2-h'^2}}{h}$	$-\left(1+\frac{1}{h^2}\right)$	$\frac{z}{h\sqrt{1-z^2}}$	$\frac{ht}{\sqrt{1+h^2t^2}}$	$A(h, h')\left\{\left(1-\frac{1}{n}\right)[\Pi(n, k, \Psi)-\Pi(n, k, \Psi_0)]-\frac{1}{n}[F(k, \Psi)-F(k, \Psi_0)]\right\}$
-1	$-h^2$	$-h'^2$	$-\frac{1}{h}$	$1-\frac{1}{h'^2}$	$\frac{\sqrt{h^2-h'^2}}{h}$	$\frac{h^2-h'^2}{h^2(h'^2-1)}$	$\frac{1}{h'}\sqrt{1-\frac{h^2-h'^2}{h^2}z^2}$	$h\sqrt{\frac{1-h'^2t^2}{h^2-h'^2}}$	$\frac{A(h, h')}{B(h, h')}[\Pi(n, k, \Psi)-\Pi(n, k, \Psi_0)]$
	$-h^2$	h'^2	$\frac{1}{\sqrt{h^2+h'^2}}$	$1-\frac{1}{h^2}$	$\frac{h}{\sqrt{h^2-h'^2}}$	$-\frac{1}{h^2-1}$	$\frac{1}{h\sqrt{1-z^2}}$	$\frac{\sqrt{h^2t^2-1}}{ht}$	$\frac{A(h, h')}{B(h, h')}\left\{\left(1+\frac{1}{n}\right)[\Pi(n, k, \Psi)-\Pi(n, k, \Psi_0)]-\frac{1}{n}[F(k, \Psi)-F(k, \Psi_0)]\right\}$

TABLE 3. Function r(t) in Eq. (18)

Parameters	$r(t)$
$c_b>0$, $\Delta<0$	$\frac{1}{2\sqrt{c_b}}\left\{\ln\left[\frac{2\sqrt{c_b}\sqrt{a_bt^4-(2a_b+b_b)t^2+(a_b+b_b+c_b)}}{1-t^2}+\frac{2c_b}{1-t^2}+b_b\right]-\ln\left[\frac{2\sqrt{c_b}\sqrt{a_bt_0^4-(2a_b+b_b)t_0^2+(a_b+b_b+c_b)}}{1-t_0^2}+\frac{2c_b}{1-t_0^2}+b_b\right]\right\}$
$c_b>0$, $\Delta>0$	$\frac{1}{2\sqrt{c_b}}\left\{\operatorname{Arsh}\frac{(2c_b+b_b)-t^2}{\sqrt{\Delta}(1-t^2)}-\operatorname{Arsh}\frac{(2c_b+b_b)-t_0^2}{\sqrt{\Delta}(1-t_0^2)}\right\}$
$c_b>0$, $\Delta=0$	$\frac{1}{2\sqrt{c_b}}\ln\left\{\frac{1-t_0^2}{1-t^2}\cdot\frac{(2c_b+b_b)-t^2}{(2c_b+b_b)-t_0^2}\right\}$
$c_b>0$, $\Delta<0$	$-\frac{1}{2\sqrt{c_b}}\left\{\arcsin\frac{(2c_b+b_b)-t^2}{\sqrt{-\Delta}(1-t^2)}-\arcsin\frac{(2c_b+b_b)-t_0^2}{\sqrt{-\Delta}(1-t_0^2)}\right\}$

Note. $a_b = mm'$; $b_b = -(m + 2mm' + m')$; $c_b = 1 + m + mm' + m'$; $\Delta = 4a_bc_b - b_b^2$.

Using the notation

$$\Phi(t)=\int_{t_0}^{t} dt/(1-t^2)\sqrt{L(t)};\quad r(t)=\int_{t_0}^{t} t\,dt/(1-t^2)\sqrt{L(t)},$$

where

$$t=-\frac{\theta-\nu}{\theta-\mu};\quad t_0=-\frac{\theta_0-\nu}{\theta_0-\mu};$$
$$L(t)=R(1-mt^2)(1-m't^2);$$
$$R=\pm 1;\ m=N/M;\ m'=N'/M';$$
$$M=\nu^2+p\nu+q;\ M'=\nu^2+p'\nu+q';$$
$$N=\mu^2+p\mu+q;\ N'=\mu^2+p'\mu+q',$$

we finally obtain the following equation from Eq. (16):

$$\mu A(h,h')(\mu-\nu)\{F(k,\Psi)-F(k,\Psi_0)\}-(\mu-\nu)^2\{\Phi(t)-r(t)\}=x\sqrt{\frac{A}{2}MM'}. \tag{18}$$

Values of the coefficients R, m, m', A(h, h'), B(h, h'), k, and n, of the variables t and z, and of the function Φ(t) are given in Table 2 ($z = \sin\Psi$, $z_0 = \sin\Psi_0$), and values of the function r(t) are given in Table 3. The coefficients μ and ν are determined from the system of equations

$$\left.\begin{aligned}2\mu\nu+p(\mu+\nu)+2q=0\\ 2\mu\nu+p'(\mu+\nu)+2q'=0\end{aligned}\right\},$$

and the coefficients p, p', q, and q', from the system

$$\left.\begin{aligned} p+p' &= \frac{4}{3}\cdot\frac{B}{A} \\ q+q'+pp' &= -\frac{1}{2}\cdot\frac{C}{A} \\ pq'+p'q &= 0 \\ qq' &= -\theta_m^2\left(\theta_m^2+\frac{4}{3}\cdot\frac{B}{A}\theta_m-\frac{1}{2}\cdot\frac{C}{A}\right) \end{aligned}\right\}.$$

For calculation of the temperature in the central section, we obtain from Eq. (17)

$$\mu A(h, h')(\mu-\nu)\{K(k)-F(k, \Psi_0)\}-(\mu-\nu)^2\{\Phi(t_m)-r(t_m)\} = \frac{l}{2}\sqrt{\frac{A}{2}MM'}, \qquad (19)$$

where the functions $\Phi(t_m)$ and $r(t_m)$ are taken from Tables 2 and 3 with the incomplete elliptic integral of the third kind $\Pi(n, k, \Psi)$ in Table 2 being replaced by the complete elliptic integral of the third kind $\Pi(n, k)$ and the value of the variable t in Table 3 being replaced by t_m.

NOTATION

I, electrical current; l, d, ω, p, length, diameter, cross-sectional area, and perimeter of wire; x, running value of wire length; α, coefficient of heat transfer; λ, coefficient of thermal conductivity; ρ, specific electrical resistance; β, γ, δ, temperature coefficients of resistance, heat transfer, and thermal conductivity; t, temperature of wire; t_b, temperature of surrounding medium; $\vartheta = t - t_b$, temperature drop; $F(k, \Psi)$, $K(k)$, incomplete and complete elliptic integrals of the first kind; $E(k, \Psi)$, $E(k)$, incomplete and complete elliptic integrals of the second kind; $\Pi(n, k, \Psi)$, $\Pi(n, k)$, incomplete and complete elliptic integrals of the third kind. Indices: b, at the temperature of the surrounding medium; 0, at 0°C.

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